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Using a rotating column of liquid on a recycled record player to measure the acceleration of gravity

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ABSTRACT: An experimental method that uses a cylindrical column of liquid that rests on the rotating turntable of a recycled record player to measure the acceleration of gravity is presented in this article. It relies on the established fact that the free surface of a rotating column of liquid takes the shape of a paraboloid of revolution that can be determined uniquely using the original height, radius and speed of rotation of the column, and the local acceleration of gravity. The method consists of measuring the coordinates of the free surface of a spinning liquid and the rate of rotation of the column, and relating them analytically to the local acceleration of gravity.

INTRODUCTION

A survey of the teaching literature indicates that four ways of measuring the acceleration of gravity are in common use in the instructional laboratory: the Atwood machine, the period of oscillation of a simple pendulum, the time it takes a ball to fall freely from rest through a known distance, and the amount by which a linear spring is deflected when a dead weight is placed on it [1-4].

A different method that is used in our laboratory is presented here. It consists of using the coordinates of the free surface of a spinning column of liquid to measure the acceleration of gravity.

A circular cylinder containing water initially at rest is made to rotate at constant speed about its vertical axis. After a very brief period of transient motion, the cylinder and the liquid in it begin to rotate as one single rigid body. It is well known that, under these circumstances, the free surface of liquid changes its shape from that of the horizontal plane it had when at rest, to that of a paraboloid of revolution [5]. In this article, it is shown how the coordinates of points located on the free surface of such a liquid can be used to measure the acceleration of gravity.

An apparatus was designed and built using recycled materials. A cylindrical container was created by taking a short cylindrical shell made of Plexiglas and capping one of its open ends with a flat sheet of Plexiglas. The capped end of the cylinder was glued onto the surface of a recycled *Clearaudio* turntable. A sketch of the resulting construction is shown in Figure 1. This combination resulted in a vertical cylindrical container of inside radius R that was fixed to a platform that could be rotated at a constant speed. Water was poured into the cylinder to fill it to a height z_0 but without causing any spillage over the edge of the container during rotation. When the turntable was set in motion and rigid-body motion of the assembly was achieved, the free surface of water changed its shape from a horizontal plane to that of a paraboloid of revolution. The coordinates of many points that lay on a parabola that was created by the intersection of the free surface of rotating liquid and a fixed vertical plane were measured. For each selected point, the measurement included its height relative to the bottom of the cylinder and its radial distance from the vertical axis of the cylinder. These coordinates were compared with those of the corresponding points on the parabola that is indicated by analysis. It was possible to determine the local acceleration of gravity using those comparisons.

This exercise was used in two different courses as a hands-on application of what was being taught and learned. It was used in Dynamics, which is offered at the sophomore level; and again in Fluid Mechanics, which students take in their Junior year. Analysis that was appropriate to each course was used to prove that the free surface of a cylindrical column of liquid in rigid-body rotation changed from a flat plane to a paraboloid of revolution, as illustrated below.

USE OF PARTICLE DYNAMICS

In this section, concepts learned in a course in College Physics and reinforced in a course on Dynamics were used to show that the rotating free surface takes the shape of a paraboloid of revolution.

When the cylinder and its contents are stationary, the free surface of liquid is horizontal, making the acceleration of gravity normal to the free surface. However, when rigid-body motion of the system has been achieved during constant rotation, a fluid particle in rotation experiences two components of acceleration. One, due to gravity, is vertically downward, as before; and the other, due to the uniform circular motion created by the rotation of the platform, is horizontally outwards and directly proportional to the distance from the axis of rotation. These two components combine to create a resultant acceleration vector, $\vec{a}$, which is normal to the new free surface at every point because it plays the role of the *pseudo* gravity. This resultant acceleration is represented in Equation (1), where ω is the rate of rotation in rad/s, r the radial distance, g the local acceleration of gravity, $\vec{\imath}$ the unit vector in the horizontal direction (positive to the right), and $\vec{k}$ the unit vector in the vertical direction (positive upwards).

$$\vec{a} = \omega^2 r\vec{\imath} - g\vec{k} \tag{1}$$

Accordingly, at every point along the free surface, the acceleration is normal to the local plane that is tangential to the free surface. This idea can be used to determine the shape of the new free surface as shown below. Given the fact that the motion has rotational symmetry, it can be studied using any vertical plane, without loss of generality. A vertical plane with a coordinate system *r z*, where *r* is the radial distance measured from the axis of the cylinder and *z* is a vertical distance measured from the bottom of the cylinder were considered. The trace in this vertical plane of the paraboloid of revolution that was generated during the rotation of the cylinder is a parabola. And, in that vertical plane, the local tangent to the parabola at an arbitrary point of coordinates (*r, z*) has a slope dz/dr and is always perpendicular to the resultant acceleration vector $\vec{a} = \omega^2 r\vec{\imath} - g\vec{k}$ at the point of tangency. It follows that the dot product between the acceleration vector and a vector along the tangent is zero. Thus,

$$(\vec{\imath}dr + \vec{k}\mathrm{dz}).(\omega^2 r\vec{\imath} - g\vec{k})=0 \tag{2}$$

from which it follows that

$$\frac{dz}{dr} = \frac{\omega^2}{g} r. \tag{3}$$

After integration, one gets the equation of the trace of the free surface in the vertical plane *zr* as

$$z = \frac{\omega^2}{2g} r^2 + z_1. \tag{4}$$

To determine the constant of integration z_1, the conservation of mass was used. If no liquid was lost during the operation, then the mass of liquid inside the cylinder before rotation equals that during rotation. For an incompressible fluid such as water, this amounts to stating that the volumes of liquid in the cylinder during and before rotation are equal. Therefore,

$$\int_0^R \int_0^z 2\pi r dz dr = \pi R^2 z_0$$

from which one gets:

$$z_1 = z_0 - \frac{(\omega R)^2}{4g}. \tag{5}$$

The equation of the trace of the free surface in the *zr* plane then becomes:

$$z = z_0 - \frac{(\omega R)^2}{2g}\left[\frac{1}{2} - \left(\frac{r}{R}\right)^2\right]$$

which, when expanded, leads to:

$$z = z_0 - \frac{(\omega R)^2}{4g} + \frac{(\omega R)^2}{2g}\left(\frac{r}{R}\right)^2. \tag{6}$$

This shows that the free surface traces a parabola in the *zr* plane.

USE OF THE NAVIER-STOKES EQUATIONS

In this section, the Navier-Stokes' equations in cylindrical coordinates are used, which are learned in the first course in Fluid Mechanics, to show that the rotating free surface takes the shape of a paraboloid of revolution.

The Navier-Stokes' equations in cylindrical coordinates are shown below:

r-direction:

$$\rho\left(\frac{\partial V_r}{\partial t}+V_r\frac{\partial V_r}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta}-\frac{V_\theta^2}{r}+V_z\frac{\partial V_r}{\partial z}\right)=\rho g_r-\frac{\partial p}{\partial r}+\mu\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (rV_r)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta}+\frac{\partial^2 V_r}{\partial z^2}\right\} \quad (7)$$

θ-direction:

$$\rho\left(\frac{\partial V_\theta}{\partial t}+V_r\frac{\partial V_\theta}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta}+\frac{V_r V_\theta}{r}+V_z\frac{\partial V_\theta}{\partial z}\right)=\rho g_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (rV_\theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial V_r}{\partial \theta}+\frac{\partial^2 V_\theta}{\partial z^2}\right\} \quad (8)$$

z-direction:

$$\rho\left(\frac{\partial V_z}{\partial t}+V_r\frac{\partial V_z}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta}+V_z\frac{\partial V_z}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V_z}{\partial \theta^2}+\frac{\partial^2 V_\theta}{\partial z^2}\right\} \quad (9)$$

where $V_r, V_\theta,$ and V_z are the velocity components of the fluid in the r, θ, and z directions, respectively, ρ is the mass density of the fluid, μ is coefficient of absolute viscosity, t is time, p is pressure, g_r, g_θ and g_z are the components of the acceleration of gravity in the r, θ and z -directions, respectively.

After rigid-body rotation has been established, flow becomes steady. Motions in the radial and vertical directions cease while motion in the circumferential direction becomes uniform. Accordingly, $V_r = 0, V_z = 0, and\ V_\theta = \omega r.$ Therefore, the Navier-Stokes' equations in the r, θ, and z -directions are simplified, respectively, to:

$$\rho\omega^2 r=\frac{\partial p}{\partial r} \quad (7a)$$

$$0=\frac{\partial p}{\partial \theta} \quad (8a)$$

$$0=-\rho g-\frac{\partial p}{\partial z} \quad (9a)$$

On the free surface, the pressure is atmospheric at every point. Consequently, the free surface is a surface of constant pressure. Thus, the pressure difference between any two points on it is zero. This leads to:

$$dp=\frac{\partial p}{\partial r}dr+\frac{\partial p}{\partial \theta}d\theta+\frac{\partial p}{\partial z}dz=0. \quad (10)$$

Utilising the partial derivatives of the pressure that were obtained in Equations (7a), (8a) and (9a), Equation (10) becomes:

$$\frac{dz}{dr}=\frac{\omega^2}{g}r, \quad (11)$$

which is the same as Equation (3), obtained earlier. One can then proceed to derive Equations (4), (5) and (6), as before.

FORMULAS FOR THE ACCELERATION OF GRAVITY

Equations (5), (6) and (14) indicate that the acceleration of gravity can be calculated from experimental data in four different ways: First, from Equation (5), a formula for calculating g is,

$$g=\frac{(\omega R)^2}{4(z_0-z_1)}\ . \quad (12)$$

Secondly, by setting $r = R$ in Equation (6), finding the corresponding height, $z_2 = z\ (r=R)$, then, solving for g, one gets:

$$g=\frac{(\omega R)^2}{4(z_2-z_0)}\,. \quad (13)$$

The measurement of z_2, the height of water on the wall of the cylinder during rotation, was not possible, because the tip of the sliding micrometer could not reach the wall of the tank due to the fact that the stem of the micrometer had a finite thickness. For this reason Equation (13) could not be used with the data that were collected.

Two other ways to determine the acceleration of gravity came from fitting a parabolic profile, such as given by Equation (14), to the experimental data obtained for the height of the free surface as a function of radial position.

$$z = a_0 + a_2 r^2 \quad (14)$$

The resulting fit allowed one to determine the coefficients a_0 and a_2, each of which could be used to determine g. Comparing Equation (6) with Equation (14), one gets one expression for the acceleration of gravity using the coefficient a_2 as shown in Equation (15),

$$g = \frac{\omega^2}{2a_2}. \quad (15)$$

Another formula for the acceleration of gravity was obtained from using the coefficient a_0, as shown in Equation (16),

$$g = \frac{(\omega R)^2}{4(a_0 - z_0)} \quad (16)$$

Collected data were used in Equation (12), (15) and (16) to determine the acceleration of gravity.

THE EXPERIMENTAL APPARATUS

A sketch of the apparatus that was used is shown in Figure 1. It consists of a fixed base onto which the turntable-and-cylinder assembly is mounted. Whereas in the experiments of Pintao and Souza Filho the water in the cylinder rotated around a very thin vertical shaft, the shaft that ordinarily held a record in place had been cut off in our experiments [1]. Two vertical posts that are mounted near the edge of that base support a horizontal beam that is graduated and to which a depth micrometer is attached. The micrometer, with graduations in 0.001 in, can be made to slide along the beam and is mounted vertically on it with its tip directed downwards in such a way that the latter can be extended or retracted, as needed, in order to reach the free surface of the liquid in the tank. The micrometer reading gave the distance between a point of the free surface and a fixed horizontal plane that had a known elevation relative to the bottom of the tank. This reading was later converted to z, the depth of liquid in the tank at a given location, using the known elevation of the plane of reference. During a given test run, the turntable was set in motion and enough time was allowed to elapse in order for the liquid to achieve rigid-body motion, before a reading was taken. Then, the micrometer was moved to the desired position along the beam and the location of the tip was adjusted until it touched the free surface of liquid. After contact with the liquid had been achieved, the reading of the micrometer was taken and recorded. Subsequently, the micrometer was moved to the next point where the adjustment and measurement processes were repeated. This sequence was continued until all selected points were tested.

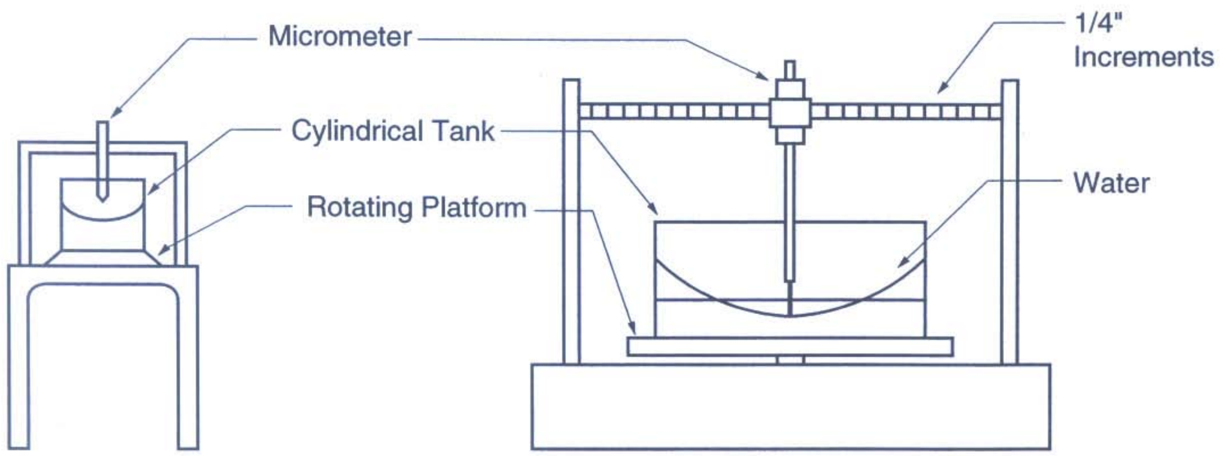

Figure 1: Experimental apparatus.

A Clearaudio turntable that had been recycled from a record player was used; it could be set to rotate at each of three different constant speeds. Each speed was measured five times using a stopwatch and the results were averaged. The averaged values were: 33.537 rpm, 45.665 rpm and 78.524 rpm, respectively. These averaged speeds were verified using a strobe light and it was found that the stopwatch measurements were accurate to the three significant figures shown above.

In these tests, the diameter of the cylinder was 6.69 in (17.53 cm) and measurements were taken at 0.25-in intervals, starting from the centre of the cylinder and moving progressively toward the wall of the cylinder, then stopping as close to that wall as the thickness of the micrometer (0.094 in) allowed. Similar measurements were made on the opposite radius in order to collect data corresponding to a diameter. A completed collection allowed the trace of the free surface of the rotating liquid onto a fixed vertical plane to be sampled. For each speed of rotation of the turntable, the height of the free surface was measured at 27 different points.

EXPERIMENTAL RESULTS

Collected data corresponding to the different speeds of the turntable are shown graphically in Figure 2, where the dashed line shows the level of water in the tank prior to the initiation of a rotation; the dots represent experimental data; and the solid curves are plots of the best parabolic fits to the data corresponding to the rotational speeds tested, as indicated by Equation (14). Equations for the parabolic fits were obtained using MATHEMATICA.

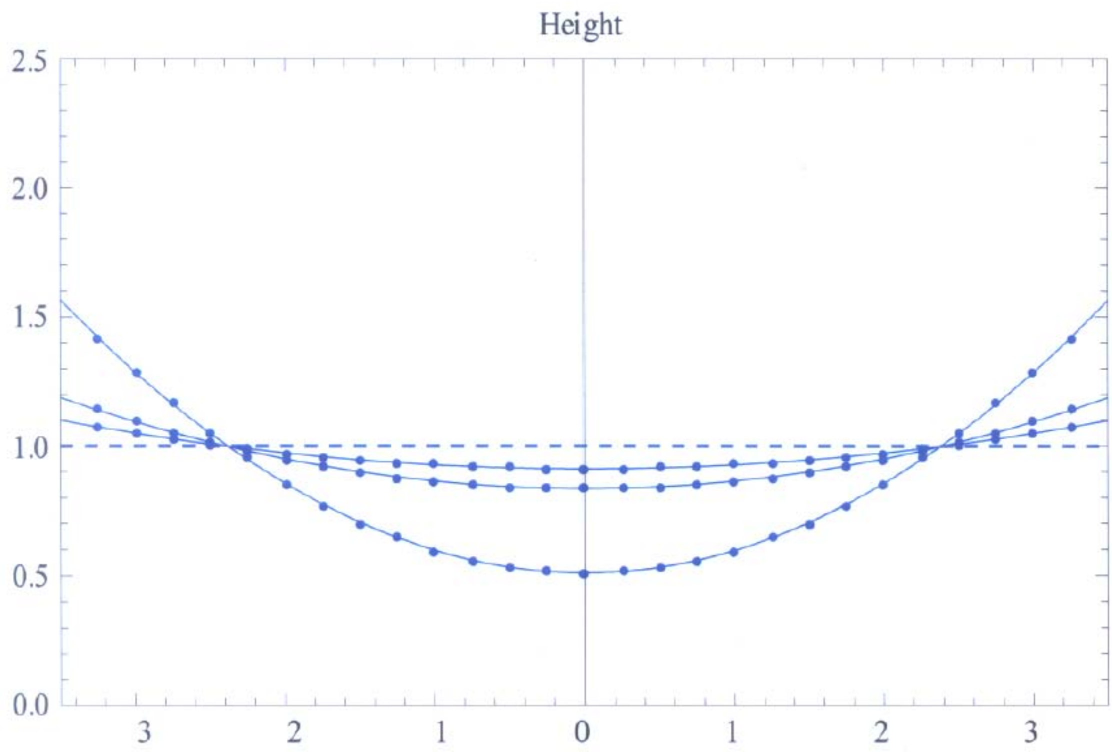

Figure 2: Superimposed free-surface profiles for ω=3.512 rad/s (highest curve on the axis, lowest on the wall); ω=4.782 rad/s (middle curve); and ω= 8.223 rad/s (lowest curve on the axis, highest on the wall).

Equations (12), (15) and (16) were used independently with the same set of data to determine the magnitude of the acceleration of gravity at the test site. The results are shown in the Tables below.

Table 1: Experimental data collected with the apparatus shown in Figure1.

Rotation speed (ω)		Original height, z_0 (from raw data)		Height at centre, z_1 (from raw data)		Coefficient a_0 (curvefit, Eq. (14))		Coefficient a_2 (curvefit, Eq. (14))	
(RPM)	(Rad/s)	(in)	(cm)	(in)	(cm)	(in)	(cm)	(in^{-1})	(cm^{-1})
33.537	3.512	1.000	2.54	0.9106	2.313	0.9108	2.313	0.0159	0.0404
45.665	4.782	1.000	2.54	0.8344	2.119	0.8346	2.120	0.0296	0.0752
78.524	8.223	1.000	2.54	0.5103	1.296	0.5110	1.298	0.0875	0.2223

Table 2: Experimental accelerations.

Rotation speed (ω)		Acceleration (using Eq. (12))		Acceleration (using Eq. (15))		Acceleration (using Eq. (16))	
(RPM)	(Rad/s)	(ft/s^2)	(m/s^2)	(ft/s^2)	(m/s^2)	(ft/s^2)	(m/s^2)
33.537	3.512	32.146	9.795	32.322	9.852	32.209	9.817
45.665	4.782	32.184	9.809	32.189	9.811	32.204	9.815
78.524	8.223	32.173	9.804	32.199	9.814	32.209	9.817

For each speed setting, the value of the acceleration was computed using the same data as in Equations (12), (15) and (16), respectively. Using 9.810 m/s^2 (32.185 ft/s^2) as the expected magnitude of the acceleration of gravity in the laboratory in Fort Wayne, Indiana, USA, latitude 41^0, where the data used in this article were collected, the discrepancies between this expected and the measured values of the acceleration were computed and presented in Table 3 [7]. Using the nine values determined, it can be seen that the largest discrepancy was found to be 0.43 %. Generally, the lowest speed of rotation used (33.537 rpm) yielded larger discrepancies. This was attributed to relative errors in measurements consequent to the shallowness of the deflected surface.

Table 3. Discrepancies between measured and expected values of the acceleration of gravity.

Rotation speed (ω)		Discrepancies (using Eq. (12))		Discrepancies (using Eq. (15))		Discrepancies (using Eq. (16))	
(RPM)	(Rad/s)	(%)	(m/s^2)	(%)	(m/s^2)	(%)	(m/s^2)
33.537	3.512	-0.121	9.795	0.426	9.852	0.073	9.817
45.665	4.782	-0.003	9.809	0.014	9.811	0.059	9.815
78.524	8.223	-0.037	9.804	0.043	9.814	0.075	9.817

CONCLUSIONS

When Equation (12) is rearranged in order to isolate the angular velocity, ω, one gets:

$$\omega^2 = B(z_0 - z_1) \tag{17}$$

where $B = \frac{4g}{R^2}$, from which it can be seen that

$$g = \frac{BR^2}{4} \tag{18}$$

The standard value for the acceleration due gravity in vacuum that was adopted by the International Committee on Weights and Measures is 980.665 cm/s^2, or 32.174 ft/s^2. The acceleration of gravity at any latitude and elevation is given by Helmert's equation [6] as:

$$g = 980.616 - 2.5928cos2\phi + 0.0069cos^2 2\phi - 3.086 \times 10^{-6} H , \tag{19}$$

Where ϕ is the latitude, H is the elevation in centimetres, and g is the acceleration of gravity in cm/s^2.

Using a Latitude of ϕ=41.08^0 , Longitude 85.11 W and an elevation of H = 23287 cm (764 ft) for Fort Wayne, Indiana, USA, where the experiments were conducted, into the Helmert's equation gives g=980.1834 cm/s^2. However, the standard value used for the computation of the discrepancies is the one found in textbooks.

Pintao and de Souza Filho [1] used Equations (17) and (18) and a computerised rotational system equipped with data acquisition to determine the acceleration of gravity from a rotating column of liquid inside a circular cylinder. They plotted ω^2, the square of the angular velocity of the tank versus the maximum downward deflection, (z_0-z_1), of the free surface of liquid during rotation and related the slope, B, of the resulting straight line to the acceleration of gravity and the inside radius of the tank, as shown in Equation (18). Their results were within 1% of the accepted value.

The discrepancies between the measured values of the acceleration of gravity using the method presented in this article and the expected value are shown in Table 3. It can be seen that they are small in all three cases. The largest magnitudes of the discrepancies were 0.15%, 0.43% and 0.08%, respectively from Equation (12), Equation (15) and Equation (16). Comparing the accelerations of gravity obtained using the three speeds, it was found that the best results were obtained consistently from the middle speed. Comparing the equations used across the three speeds, the best results were obtained from Equation (16). In all cases, these results compare well with those obtained by Pintao and de Souza Filho [1].

This method offers an alternative way to measure the acceleration of gravity using equipment that can be made by students, mostly from recycled materials and commonly available equipment such as stopwatches, Plexiglas tubes, recycled turntables from old record players, micrometers, strong adhesive materials and metal saws. As the world considers reducing the carbon footprint of human activities on the environment, the wide availability of DVD players and other advanced technologies offers engineering faculty an opportunity to find ways to make creative uses of otherwise obsolete components of machines and devices in their instructional and research laboratories. Albeit in small a way, this experiment demonstrates that this can be done successfully.

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